2.17.1. Semantic Problems:
Tautology, Contradiction, and Logical Equivalence

A. **Translate** each English sentence into the formal language and build a **truth table** for that formal sentence. On the basis of that truth table, find a **simpler formal sentence** that's **logically equivalent** to the original; then **translate** that simpler sentence back **into English**.

1. Neko will either eat fish or eat fish.

2. Letitia's not a goth, and she's also not a goth.

3. Jack is a cat who either scales cliffs or doesn't.

4. Either Jack is a cat who scales cliffs, or he's one who doesn't.

5. Either Jake will attend Logicpalooza, or both he and Jezebel will.

6. Either Jake will attend Logicpalooza, or either he or Jezebel will.

7. Lucretia went out, and she did so without taking her umbrella.

8. Elvis is a gambler, but one who doesn't drink whiskey.

9. Time gliding is either possible or impossible, but not both.

10. Gasoline fights are either dangerous or pleasant, and they're also either dangerous or unpleasant.

11. Either Suki is neither tall nor tall and skinny, or she's neither tall nor skinny.

12. Unless mahjongg's neither illegal nor illegal, it's either legal or legal.

13. Either Trixie finished her homework, or she went to the party without finishing it.
    (*Note: the simpler sentence won’t appear earlier in the truth table.*)

14. Letitia cooked an omelet, but she didn’t do so without breaking some eggs.[1]
    (*Note: the simpler sentence won’t appear earlier in the truth table.*)

B. For each pair of sentences, **translate** both into formal language and build **truth tables** for them to show that the two sentences are **logically equivalent**.

1a. Neither Trixie nor Elvis failed to attend the poker tournament.
1b. Both Trixie and Elvis attended the poker tournament.

2a. Letitia and Lucretia didn’t both fail to attend goth night at Novo.
2b. Either Letitia or Lucretia attended goth night at Novo.

3a. We’re having truffles, and either grog or grappa.
3b. We’re having either truffles and grog or truffles and grappa.

4a. Either we’re having truffles or we’re having grog and grappa.
4b. We’re having either truffles or grog, and either truffles or grappa.

5a. Neko is a cat who likes fish, and one who also likes cream cheese.
5b. Neko is a cat who likes both fish and cream cheese.

6a. Kitty’s not a singer who won a voice scholarship.[2]
6b. Kitty’s not a singer unless she’s one who didn’t win a voice scholarship.

7a. We won’t have a day off unless there’s a good reason.
7b. We won’t have a day off without there being a good reason.[1]

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[1] On negated “without” sentences, see 2.10 §3; on the intermittent parallel between “unless” and “without,” see 2.10.1 B.
[2] On the negation of a sentence containing a relative clause, see 2.10 §3.

C. For each numbered group of sentences below, use truth tables to show that all the sentences in that group are **logically equivalent**.

Distribution

1a. $(P \wedge (Q \vee R)$

1b. $((P \wedge Q) \vee (P \wedge R))$

2a. $(P \vee (Q \wedge R))$

2b. $((P \vee Q) \wedge (P \vee R))$

Idempotence

3a. $(P \vee P)$

3b. $(P \wedge P)$

3c. P

Absorption

4a. $(P \vee (P \wedge Q))$

4b. $(P \wedge (P \vee Q))$

4c. P

D. For each of the following claims about **consistency and inconsistency**, state whether that claim is **true or false**.

1. If a set containing just one sentence is consistent, then that sentence is consistent.

2. If a set containing just one sentence is inconsistent, then that sentence is inconsistent.

3. If a sentence is consistent, then every set of sentences containing that sentence is consistent.

4. If a sentence is inconsistent, then every set of sentences containing that sentence is inconsistent.

5. If a set of sentences is consistent, then every sentence in that set is consistent.

6. If a set of sentences is inconsistent, then every sentence in that set is inconsistent.

7. If a set of sentences is inconsistent, then at least one sentence in that set is inconsistent.

E. **Translate** each of the following sentences into formal language; then use a **truth table** to decide whether that sentence is a **tautology**, a **contradiction**, or **neither**.

1. Dr. Slim is a weasel who isn’t a weasel.

2. Jack either scales cliffs or doesn’t.

3. Jack is a cat who either scales cliffs or doesn’t.

4. Unless we’re having both truffles and grog, we’re not having truffles.

5. Unless we’re not having truffles, we’re having either truffles or grog.

6. Suki passed Business Logic without studying for the Business Logic exam, though she did study for the Business Logic exam.

7. Unless Rex’s handwriting is illegible, it’s either legible or beautiful.

8. Unless Rex’s handwriting is illegible, it’s both legible and beautiful.

9. Although Rex’s handwriting is illegible, it’s both legible and beautiful.

10. Unless Dr. Slim is a physician, he’s not a physician who performs surgery.[3]

11. Unless Dr. Slim is a physician who performs surgery, he’s not a physician.

12. Dick and Dora both ordered a Sloe Gin Fizz, unless neither of them did.

13. Either Dick ordered a Sloe Gin Fizz or Dora ordered one – unless neither of them did.

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[3] On the negation of a sentence containing a relative clause, see 2.10 §3.

F. Following a break-in at the Neptuna Seafood Lounge, various people testify about whether Neko or Jack were in the neighborhood of the crime scene. Decide for each person **whether they gave consistent testimony**. (Use the **same translation key** for everyone's testimony; leave out words in parentheses.)

*(Tip: since several people may use the same sentence, **build truth tables for all the sentences appearing here**, then decide on each person's consistency.)*

1. **Neko**: "Me and Jack weren't *both* in the neighborhood. (In particular,) I wasn't in the neighborhood."

2. **Jack**: "Neither of us were in the neighborhood."

3. **Jake**: "Neko wasn't in the neighborhood unless Jack was. (Now, as a matter of fact) Neko was in the neighborhood, but Jack wasn't."

4. **Barbie**: "Neko and Jack weren't *both* in the neighborhood. (But) Neko wasn't in the neighborhood unless Jack was."

5. **Trixie**: "Neko and Jack weren't *both* in the neighborhood. (But) Neko wasn't in the neighborhood unless Jack was. (In fact,) Jack was in the neighborhood."

6. **Elvis**: "Neko and Jack weren't *both* in the neighborhood. (But) Neko wasn't in the neighborhood unless Jack was. (In fact,) Neko was in the neighborhood."

7. **Rex**: "Either Neko and Jack were both in the neighborhood or neither of them were."

8. **Suki**: "Either Neko and Jack were both in the neighborhood or neither of them were. (In fact,) Neko was in the neighborhood."

9: **Dr. Slim**: "Either Neko and Jack were both in the neighborhood or neither of them were. (Now in fact,) Neko was in the neighborhood, but Jack wasn't."

10. **Jezebel**: "Either Neko and Jack were both in the neighborhood or neither of them were. (Moreover,) either Neko was in the neighborhood or she wasn't. (Now in fact,) Neko was in the neighborhood, but Jack wasn't."

G. Use the testimony from **(F)** to **answer** the following questions.

1. Is **Neko’s** testimony **consistent** with **Jack’s**?

2. Is **Trixie’s** testimony **consistent** with **Suki’s**?

3. Is **Trixie’s** testimony **consistent** with **Neko’s**?

4. Is **Barbie’s** testimony **consistent** with **Nekos**?

5. Is **Barbie’s** testimony **consistent** with **Trixie’s**?

6. Is **Barbie’s** testimony **consistent** with **Elvis’s**?

7. Is **Barbie’s** testimony **consistent** with **Jack’s**?

8. Is **Trixie’s** testimony **consistent** with **Jack’s**?

9. Are **Barbie’s**, **Neko’s** and **Jack’s** testimony all **consistent** together?

10. Are **Barbie’s**, **Neko’s**, and **Trixie’s** testimony all **consistent** together?

H. Once we see that the sentence “(P ∨ ~P)” is a tautology, we see as well that “(Q ∨ ~Q)” is a tautology, that “((P ∧ Q) ∨ ~(P ∧ Q))” is, and so on. Say that an **instance** of a sentence S is the result of **replacing a sentence letter** in sentence S **with some formal sentence**. So (2) and (3) are each an instance of Sentence (1).

(1) (P ∨ ~P)
(2) (**Q** ∨ ~**Q**) *(replacing “P” with “Q”)*
(3) (**(P ∧ Q)** ∨ ~**(P ∧ Q)**) *(replacing “P” with “(P ∧ Q)”)*

Then it seems correct to say: since (1) is a tautology, any instance of (1) will also be a tautology. And in general: **if a sentence is a tautology, then every instance of that sentence is a tautology**.

But if in (1) we replace **only the first occurrence** of "P" with a sentence, we get Sentence (4), which isn't a tautology.

(1) $(\underline{\mathbf{P}} \lor \sim\mathbf{P})$
(4) $(\underline{\mathbf{Q}} \lor \sim\mathbf{P})$

And if we replace every "P" in (1), but **replace** different occurrences of "P" with **different sentences**, we get Sentence (5), which isn't a tautology.[4]

(5) $(\mathbf{Q} \lor \sim\mathbf{R})$

1. State conditions for being an instance of a sentence which give the right results – namely, that **any instance of a tautology is a tautology**.[5] In particular: what **conditions** needs to be added on **how we replace a sentence letter with a sentence**, in order to block (4) and (5) from counting as instances of (1)?[6]

2. According to the definition of "instance" from **Problem (1)**, is it also true that **every instance of a contradiction is a contradiction**?

3. Will every **instance of a consistent sentence** (a **non-contradiction**) likewise be a consistent sentence?

4. Will every **instance of a non-tautology** likewise be a non-tautology?

5. If sentence ● is an instance of sentence ▲, is ▲ guaranteed to be an instance of ●?

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[4] In fact if we replace the left "P" in (1) with "$(P \land \sim P)$" and the right "P" with "$(P \lor \sim P)$," we get a contradiction: "$((P \land \sim P) \lor \sim(P \lor \sim P))$".

[5] Compare the conditions on being an instance with those on being an **instance of a formula** discussed in 5.8.

[6] Such systematic replacement is sometimes called "uniform substitution".

I. A Puzzle about Disjunctions and Conjunctions. We've been counting "unless" as a disjunction phrase, translated by the vel.

P: Rex will go **Q**: Barbie (will) go

1. Rex won't go **unless** Barbie goes. $(\sim P \vee Q)$

But note that "unless" can appear accompanied by the words "too" or "as well".

2. Rex won't go **unless** Barbie goes **too**.
3. Rex won't go **unless** Barbie goes **as well**.

We noted earlier that "too" and "as well" typically accompany conjunction phrases.

4. The movie is entertaining, **and** it's informative **too**.
5. Kids will enjoy the movie, **but** adults will like it **as well**.

That suggests that Sentences (2) and (3) should instead be translated with a **conjunction as its right part** (meaning "Rex won't go unless both he and Barbie go").

Rex won't go **unless** Rex goes and Barbie goes **too**. $(\sim P \vee \underline{(P \wedge Q)})$
Rex won't go **unless** Rex goes and Barbie goes **as well**. $(\sim P \vee \underline{(P \wedge Q)})$

Build **truth tables** for "$(\sim P \vee Q)$" and "$(\sim P \vee (P \wedge Q))$" to show why **there's no semantic reason to make this change** – so that, for purposes of truth and validity, we can continue translating sentences (2) and (3) as simply "$(\sim P \vee Q)$".

J. Build a truth table for each of the following sentences, to show that the sentence is a **tautology**.

T 2.1. $\sim(P \wedge \sim P)$

T 2.2. $(P \vee \sim P)$

T 2.3. $\sim(P \wedge \sim(P \vee P))$

T 2.4. $(P \vee \sim(P \vee P))$

T 2.5. $(P \vee (\sim P \vee \sim P))$

T 2.6. $(P \vee (\sim P \vee Q))$

T 2.7. $(\sim Q \vee (\sim P \vee Q))$

T 2.8. $((P \vee Q) \vee \sim(P \wedge Q))$

T 2.9. $((P \vee Q) \vee (\sim P \wedge \sim Q))$

T 2.10. $((\sim(P \vee Q) \vee \sim(\sim P \vee Q)) \vee Q)$

T 2.11. $(((P \wedge Q) \vee (\sim P \wedge Q)) \vee \sim Q)$

T 2.12. $(((P \vee Q) \wedge (\sim P \vee Q)) \vee \sim Q)$

T 2.13. $\sim(((P \wedge Q) \vee (\sim P \wedge Q)) \wedge \sim Q)$

T 2.14. $\sim(((P \vee Q) \wedge (\sim P \vee Q)) \wedge \sim Q)$

T 2.15 $\sim((\sim P \vee \sim P) \wedge (P \vee (P \wedge Q)))$

T 2.16. $(((P \wedge Q) \vee (\sim P \wedge Q)) \vee ((P \wedge \sim Q) \vee (\sim P \wedge \sim Q)))$